Semidefinite relaxations for optimal control problems with oscillation and concentration effects

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July 8, 2014

The bounded control case

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What if $U = \mathbb{R}^m$?

Example: concentration effects

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\begin{aligned}
J = \inf \quad & \int_0^1 (t - \frac{1}{2})^2 u \, \mathrm{d}t \\
\text{s.t.} \quad & \dot{y} = u, \\
& y(0) = 0, \quad y(1) = 1 \\
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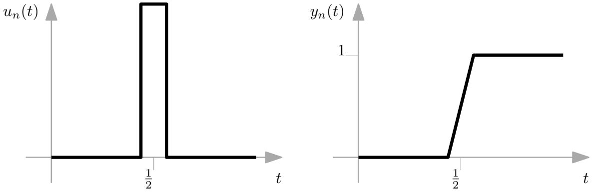
Minimizing sequence:



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$$J = \inf \int_0^1 \left(\frac{u^2}{1+u^4} + (y-t)^2 \right) \mathrm{d}t$$

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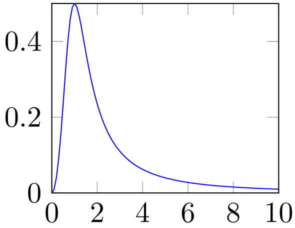

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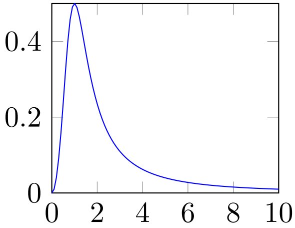

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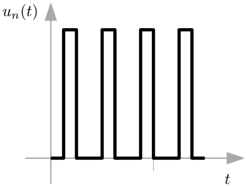

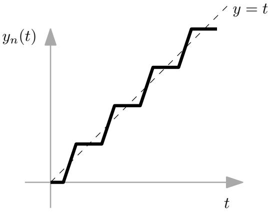

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$\Longrightarrow$ extends [Kružík, Roubíček: '98] for non-convex problem.

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Theorem (DiPerna and Majda)

For bounded $\{u_k\}_{k\in\mathbb{N}}$ *in* $L^p([t_0, t_f]; \mathbb{R}^m)$, $\exists$ *subsequence,* $\sigma \in \mathcal{M}^+([t_0, t_f])$ *and* $\nu(\mathrm{d}\bar{u}|t) \in \mathcal{P}(\gamma\mathbb{R}^m)$ *defined* σ*-a.e. such that for any* $g \in C([t_0, t_f])$ *and any* $w \in \mathcal{R}$*:*

$$\lim_{k\to\infty} \int_{t_0}^{t_f} g(t)v(u_k(t))\mathrm{d}t \; = \int_{t_0}^{t_f} \int_{\gamma\mathbb{R}^m} g(t)w(\bar{u})\nu(\mathrm{d}\bar{u}|t)\sigma(\mathrm{d}t) \; ,$$

where $v(\bar{u}) = w(\bar{u})(1 + |\bar{u}|^p)$.

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$$\lim_{k\to\infty} \int_{t_0}^{t_f} g(t)v(u_k(t))\mathrm{d}t \;=\int_{t_0}^{t_f}\int_{\gamma\mathbb{R}^m} g(t)w(\bar{u})\nu(\mathrm{d}\bar{u}|t)\sigma(\mathrm{d}t)\ ,$$

where $v(\bar{u}) = w(\bar{u})(1+|\bar{u}|^p)$.

Example: to $u \in L^p$ corresponds $\sigma = (1+|u(t)|^p)\,\mathrm{d}t$ and $\nu = \delta_{u(t)}(\mathrm{d}\bar{u}|t)$.

Relaxed OCP

$$J = \min_u \int_{t_0}^{t_f} h(t, y(t), u(t)) \, \mathrm{d}t$$

$$\text{s.t. } \dot{y} = f(t, y(t), u(t))$$

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Relaxed as:

$$J_r = \min_{\sigma,\nu} \int_{t_0}^{t_f} \int_{\gamma\mathbb{R}^m} \frac{h(t, y(t), \bar{u})}{1 + |\bar{u}|^p} \nu(\mathrm{d}\bar{u}|t)\sigma(\mathrm{d}t)$$

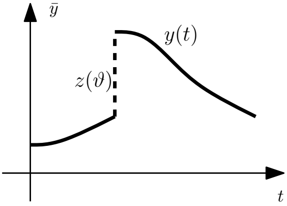
$$\text{s.t. } \dot{y} = \int_{\gamma\mathbb{R}^m} \frac{f(t, y(t), \bar{u})}{1 + |\bar{u}|^p} \nu(\mathrm{d}\bar{u}|t)\sigma,$$

$$(\sigma, \nu) \in \mathcal{DM}^p([t_0, t_f]; \mathbb{R}^m)$$

See [Kružík, Roubíček: '98].

Occupation measures

Fix admissible (σ, ν, y):



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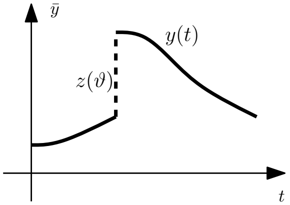

$$\xi(B|t) := \begin{cases} \delta_{y(t)}(B) & \text{if } t \notin J \\ \int_0^{d_t} I_B(z_t(\vartheta))/d_t \, \mathrm{d}\vartheta & \text{otherwise} \end{cases}$$

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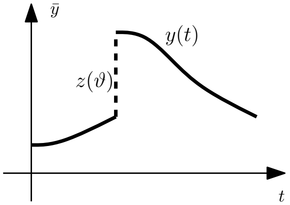

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$$\mu(\mathrm{d}t\mathrm{d}\bar{y}\mathrm{d}\bar{u}) := \xi(\mathrm{d}\bar{y}|t)\, \nu(\mathrm{d}\bar{u}|t)\, \sigma(\mathrm{d}t)$$

Weak ODE integration

Test μ with $v \in C^1(T \times Y)$ along trajectories of admissible.

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Proposition

$$v(t_f, x(t_f^+)) - v(t_0, x(t_0^-)) = \langle \frac{\partial v}{\partial t} \frac{1}{1 + |\bar{u}|^p} + \frac{\partial v}{\partial \bar{y}} \frac{f(t, \bar{y}, \bar{u})}{1 + |\bar{u}|^p}, \mu \rangle$$

The measure LP

Convex relaxation:

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\begin{aligned}
J_{meas} = \inf_{\mu} \ & \langle \frac{h(t,\bar{y},\bar{u})}{1+|\bar{u}|^p}, \mu \rangle \\
\text{s.t. } & \forall v \in C^1(T \times Y) : \\
& v(t_f, y(t_f^+)) - v(t_0, y(t_0^-)) = \langle \frac{\partial v}{\partial t} \frac{1}{1+|\bar{u}|^p} + \frac{\partial v}{\partial \bar{y}} \frac{f(t,\bar{y},\bar{u})}{1+|\bar{u}|^p}, \mu \rangle \\
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Compare this with [Vinter, Lewis: SICON '78] or [Vinter: SICON '93].

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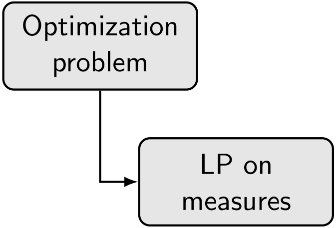
The moment approach



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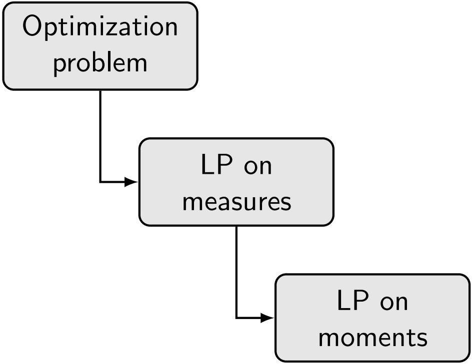

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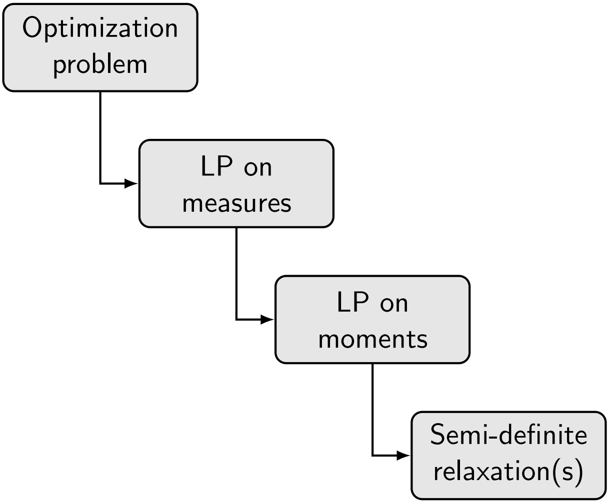

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- Let $\mathbf{X} := \{x \in \mathbb{R}^n : g_i(x) \geq 0, \quad i = 1, ..., m\}$

Theorem (Putinar)

$$\mu \in \mathcal{M}^+(\mathbf{X}) \text{ } \textit{iff:}$$

$$M(z) \succeq 0, \qquad M(g_i * z) \succeq 0 \quad \forall i$$

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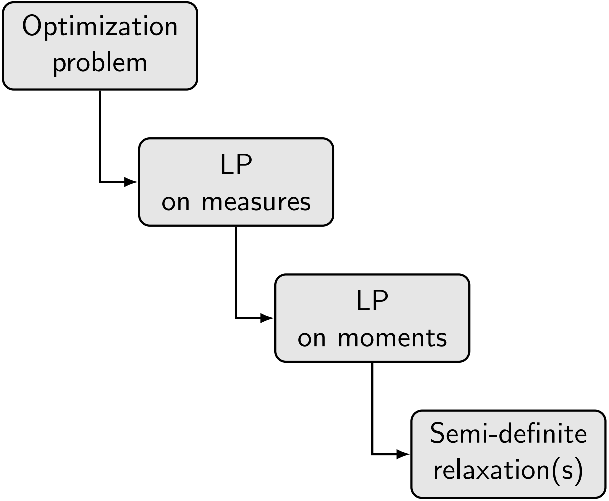

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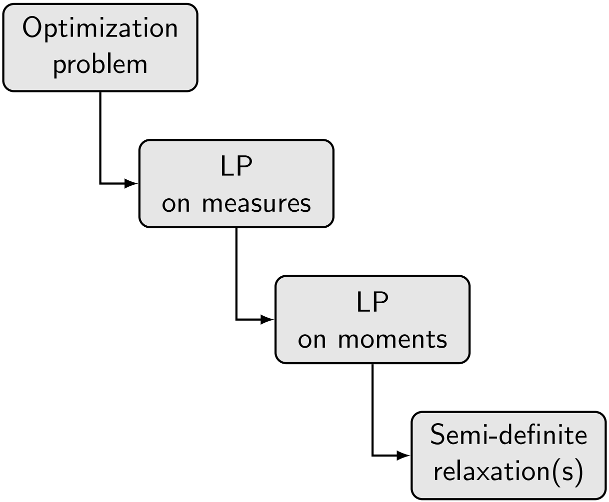
`GloptiPoly` [Henrion et al.]: toolbox for automatic generation of the SDP relaxations from the measure LP.

The moment approach



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4. Approximate support = non-zero atoms.

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$$\left(L_z(t^k)\right)_{k\in\mathbb{N}} = (2.0000,\ 1.0000,\ 0.5833,\ 0.3750,\ 0.2625,\ 0.1979,\ \ldots),$$

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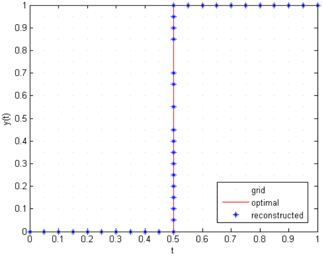

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$$\left(L_z(t^k)\right)_{k\in\mathbb{N}} = (2.0026,\ 1.0026,\ 0.6692,\ 0.5026,\ 0.4026,\ 0.3359,\ \ldots),$$

$$\left(\langle t^k, \sigma^*\rangle\right)_{k\in\mathbb{N}} = (2.0000,\ 1.0000,\ 0.6667,\ 0.5000,\ 0.4000,\ 0.3333,\ \ldots),$$

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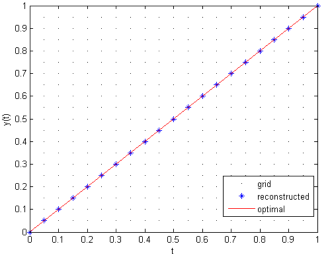

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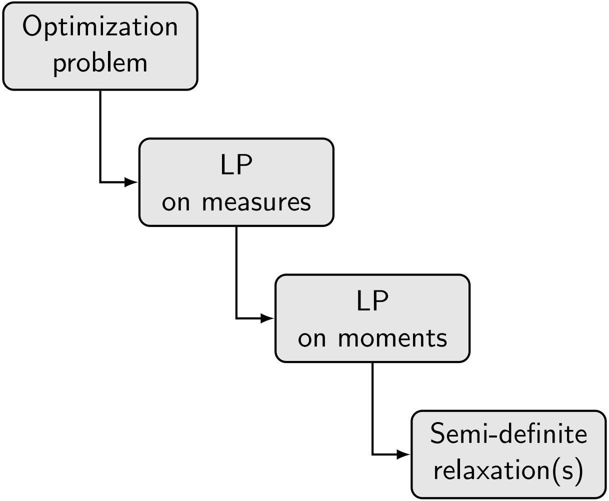
The moment approach



Highlights of the method

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- Global resolution, even for non-convex problems.
- Easy handling of state constraints.
- Straightforward to implement in `GloptiPoly`.
- Currently $n + m \leq 5$, but SDP solvers are getting faster (`Mosek`) ...

Thanks!

- Presentation and paper version available at

  `http://mathclaeys.wordpress.com`

- Mini-course on polynomial optimization: D. Henrion (Th. 10:30, A901), M. Putinar (Th. 11:30, A901), MC (Fr. 10:30, A901), M. Korda (Fr. 11:30, A901).

- This research was supported by the AVČR-CNRS project "Semidefinite programming for nonconvex problems of calculus of variations and optimal control".